

Non-Extensive Approach to Quark Matter

Tamás S. Biró¹, Gábor Purcsel¹ and Károly Ürmösy¹

KFKI Research Institute for Particle and Nuclear Physics

Received: date / Revised version: date

Abstract. We review the idea of generating non-extensive stationary distributions based on abstract composition rules of the subsystem energies, in particular the parton cascade method, using a Boltzmann equation with relativistic kinematics and modified two-body energy composition rules. The thermodynamical behavior of such model systems is investigated. As an application hadronic spectra with power-law tails are analyzed in the framework of a quark coalescence model.

PACS. 21.65.Qr quark matter – 25.75.Ag global features in relativistic heavy ion collisions – 05.20.Dd kinetic theory

1 Introduction

Power-law tailed distributions are abundant in Nature and in human technology ranging from high energy particle spectra to fluctuations in stock markets or connectivity statistics in the Internet. It would be natural to explain this abundance by a universal, statistical limiting distribution since different causes result in similar outcomes. A more prestigious attempt is to set such phenomena into a united framework of non-extensive thermodynamics, based on certain generalizations of familiar basic formulas. In particular generalizations of the Boltzmann – Gibbs – Shannon entropy formula were sought as funding stones for such a general treatment [1, 2, 3, 4, 5, 6, 7, 8].

Several basic questions arise during this enterprise: among those the uniqueness of equilibrium state and the entropy function describing irreversibility, the connection between composition rules for basic thermodynamical quantities between two large subsystems and the extensivity limit for a system with a large number of degrees of freedom, and the very question that which microscopical mechanisms lead to such a distribution. Is this a sign of non-equilibrium, of incomplete equilibrium or just of a new, generalized kind of equilibrium? Applying and justifying a statistical, least thermodynamical approach to high energy heavy ion collisions, as it is being central in the experimental quest for quark matter, in particular requires clarification of the above questions. Any inference to a thermal state and a physical temperature of the quark matter from single particle spectra must connect the fit parameter measuring the spectral slope to basic principles of thermodynamics.

In recent years we have been succeeding towards answers to the above problems. After facing the fact that transverse momentum spectra fit well to a cut power-law distribution towards much higher values than just the

simple Gibbs-exponential, a particular parton cascade approach was suggested in Ref.[9] for generating these distributions. It has been observed that the stationary distribution generated and maintained by a Boltzmann type equation is intimately related to the energy composition rule used in two-particle encounters. A simple modification of the kinetic energy addition rule among two partners, which in high energy collisions is probably related to the relativistic kinematics, leads to the observed result. A general treatment of abstract composition rules is presented in Ref.[10], where the non-extensivity property is related to the deformation of addition rule and hence to the deformation of the classical Gibbs exponential.

The physics' question to begin with is the source of non-extensivity, especially for the two most relevant quantities, energy and entropy. It is relatively easy to construct examples with non-extensive energy, whenever the interaction retains a fractal structure in the phase space and therefore cannot be neglected in the large volume – large particle number limit, as it is traditional in classical thermodynamics. It is much harder to understand non-extensive entropy, however. In order to shed some light to possible mechanisms by which non-extensivity in one-particle variables, like entropy and energy, can occur in physical systems, let us investigate a very particular case.

We assume that in an N-particle system there are two-particle correlations left and seek for their relative contributions to total energy and entropy. For the sake of demonstration we regard the following special form of the two-particle density:

$$\rho_{12} = f(p_1)f(p_2)g(r_{12}), \quad (1)$$

which is factorizing in the momentum space via one-particle distribution functions, but is connected in the coordinate space via the pair-distribution function, $g(r)$, of the relative coordinates.

The trace over states is determined via phase space integrals, normalized to satisfy the following conditions in d spatial and momentum dimensions:

$$\begin{aligned} \int d^d r \, 1 &= V, \\ \int d^d r \, g(r) &= V_{\text{eff}}, \\ \int \frac{d^d p}{(2\pi\hbar)^d} f(p) &= \bar{n}, \end{aligned} \quad (2)$$

with V being the total volume, V_{eff} the available volume for a partner of a given particle, and \bar{n} the average (mean) density in the system. We normalize the integrals so that $\bar{n}V = N$ and $\bar{n}V_{\text{eff}} = N - 1$.

Under the above conditions this particular two-particle density is normalized to $\text{Tr} \rho_{12} = N(N - 1)$. The partial trace over the second particle leads to the familiar one-particle distribution function used in kinetic theories:

$$\rho_1 = \text{Tr}_2 (\rho_{12}) = (N - 1)f(p_1). \quad (3)$$

The total entropy of a correlated pair in matter, $-\text{Tr} (\rho_{12} \ln \rho_{12}) / \text{Tr} (\rho_{12})$, is expressed by

$$S_2 = - \int \frac{d^d p_1}{(2\pi\hbar)^d} \int \frac{d^d p_2}{(2\pi\hbar)^d} \int d^d r_1 \int d^d r_2 \frac{\rho_{12} \ln \rho_{12}}{N(N - 1)}. \quad (4)$$

In calculating this quantity two further individual integrals occur:

$$\begin{aligned} \bar{s} &= - \int \frac{d^d p}{(2\pi\hbar)^d} f(p) \ln f(p), \\ V_{\text{info}} &= - \int d^d r \, g(r) \ln g(r). \end{aligned} \quad (5)$$

Using these notations one arrives at:

$$S_2 = 2 \frac{\bar{s}}{\bar{n}} + \frac{V_{\text{info}}}{V_{\text{eff}}}. \quad (6)$$

Generalizing the above expression valid for the two-particle density, ρ_{12} , to an N -particle density, $\rho_{12} \dots_N$, factorized into $N(N - 1)/2$ pair contributions we obtain the following entropy per particle:

$$\frac{S_N}{N} = \frac{\bar{s}}{\bar{n}} + \frac{\bar{n}}{2} V_{\text{info}}. \quad (7)$$

The entropy of such a system is considered to be extensive, as long as the specific ratio remains finite in the large particle number limit:

$$\lim_{N \rightarrow \infty} \frac{S_N}{N} < \infty. \quad (8)$$

In this sense dangerous pair distributions are those, for which V_{info} increases with N at fixed mean density \bar{n} . In a familiar piece of matter the pair distribution function $g(r)$ approaches one at large distances, in these cases V_{info} is finite and hence the entropy is extensive. In case of a quark gluon plasma, however, some infrared magnetic modes remain non-perturbative and hence long range correlations remain. As a consequence $-g \ln g$ may not tend

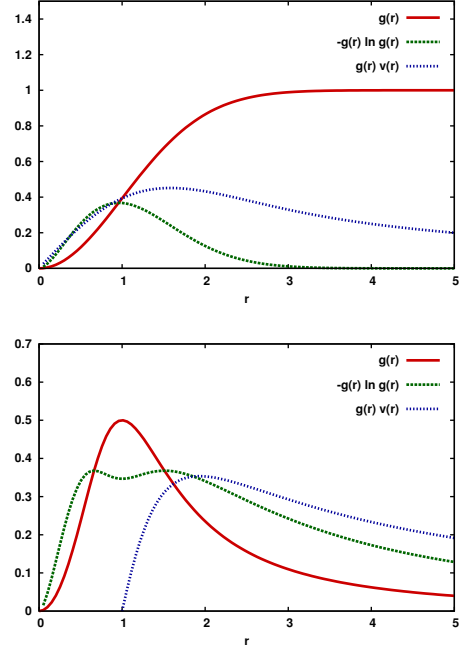


Fig. 1. Schematic plots of the pair distribution function and the corresponding energy and entropy contributions in short (top) and long range (bottom) correlated matter assuming a $1/r$ and a σr type pair potential, respectively.

to zero fast enough and therefore the integral V_{info} in eq.(5) may increase as a function of N . For example considering power-law type pair distribution functions, like $g(r) = r^a / (1 + r^b)$, the corresponding integrals up to a large radius, R scale like $V_{\text{eff}} \sim R^{d+a-b}$ and like $V_{\text{info}} \sim R^{d+a-b} \ln R$. In this case the specific entropy for large N becomes

$$\frac{S_N}{N} \longrightarrow \frac{\bar{s}}{\bar{n}} + \text{const.}(a - b)N \ln N \quad (9)$$

with some unspecified constant. For $a \neq b$ this would lead to a non-extensive entropy. For other possible sources of non-extensive entropy see Ref.[11].

The total energy can be calculated in a similar way. Assuming a $v(r_{12})$ pair-potential depending on the relative coordinate only and individual kinetic energies, $K(p_i)$, we arrive at

$$\frac{E_N}{N} = \frac{\bar{e}}{\bar{n}} + \frac{\bar{n}}{2} V_{\text{pair}} \quad (10)$$

with

$$\begin{aligned} \bar{e} &= \int \frac{d^d p}{(2\pi\hbar)^d} f(p) K(p), \\ V_{\text{pair}} &= \int d^d r \, g(r) v(r). \end{aligned} \quad (11)$$

Some typical $g(r)$ functions are shown in Fig.1. On the top figure a pair distribution function tending to one at large distances and a Coulomb-like pair potential, while on the bottom figure a linear confining potential, $v(r) \sim$

r and a power-law tailed pair distribution function are assumed.

In most physical systems studied traditionally in thermodynamics, like gases, liquids, plasmas, etc. the function $g(r)$ approaches the value one at large distances. Therefore there are no non-extensive contributions to the entropy per particle. For the energy the situation is different so far, since $v(r) \sim r^{-b}$ might not approach zero for large distances fast enough, producing this way a contribution to the energy per particle, E_N/N which may even diverge in the large N limit. Such a case is an unscreened $1/r$ -like potential in three dimensions.

2 Additivity, extensivity and abstract composition rules

Non-extensive quantities, whose amount per particle is not finite in the thermodynamic limit, are also not additive, because the repeated composition by simple addition rules always leads to a result proportional to the number of steps. It is possible, however, that one is able to find another quantity, a certain function of the non-additive one, which is additive. This way the non-extensive thermodynamics can be treated by mathematical algorithms which were designed for additive composition rules. In this section we analyze the mathematical background of composing energy and/or entropy of subsystems and then repeating this composition. The thermodynamical limit is appointed to the infinite repetition of the composition with an infinitesimal amount [10].

2.1 General rules and thermodynamical limit

Let us denote an abstract pairwise composition rule by the mapping $(x, y) \rightarrow h(x, y)$. The important question arises, that what happens if we repeat such a composition rule arbitrarily long, each time applying to an infinitesimal amount: This way one deals with the thermodynamical limit of composition rules corresponding to the energy or the entropy. The effective rule in this limit, which applies to results of repeated rules themselves, has special properties then.

From the starting rule we demand only a trivial property: that the composition with zero should be the identity

$$h(x, 0) = x. \quad (12)$$

We do not assume in general symmetry (commutativity), such as $h(y, x) = h(x, y)$ nor we demand associativity

$$h(h(x, y), z) = h(x, h(y, z)). \quad (13)$$

Here we note that the general solution of the associativity equation (13) is given by

$$h(x, y) = X^{-1}(X(x) + X(y)) \quad (14)$$

with $X(x)$ being a strict monotonic function [12]. We shall refer to this mapping function as the "formal logarithm",

because it maps the arbitrary composition rule $h(x, y)$ to the addition by taking the X -function of eq.(14):

$$X(h(x, y)) = X(x) + X(y). \quad (15)$$

Due to this construction the generalized analogs to classical extensive (and additive) quantities are their formal logarithms, whenever the composition rule is associative. As a consequence stationary distributions, in particular those obtained by solving generalized Boltzmann equations [9], are the Gibbs exponentials of the formal logarithm,

$$f(x) = \frac{1}{Z} e^{-\beta X(x)}. \quad (16)$$

Let us now regard a large number of iterations, N , of a general composition rule. We apply it to a small amount y/N and repeat this $(N - 1)$ times, constructing this way the quantity

$$x_N(y) := \underbrace{h \circ \dots \circ h}_{N-1} \left(\frac{y}{N}, \dots, \frac{y}{N} \right). \quad (17)$$

We consider the large- N limit,

$$\lim_{N \rightarrow \infty} x_N(y) < \infty, \quad (18)$$

if this is finite for a finite y , we can apply all formulas of classical thermodynamics usually applied to extensive quantities. Such a limiting quantity is extensive, but not necessarily additive. Our goal is to obtain the asymptotic composition rule,

$$x_{N_1+N_2} = \varphi(x_{N_1}, x_{N_2}) \quad (19)$$

in the limit $N_1, N_2 \rightarrow \infty$. The recursion for the n -th step of this repetitive composition is given by

$$x_n = h \left(x_{n-1}, \frac{y}{N} \right), \quad (20)$$

starting with $x_0 = 0$. Subtracting $x_{n-1} = h(x_{n-1}, 0)$ from both sides we arrive at

$$x_n - x_{n-1} = h(x_{n-1}, \frac{y}{N}) - h(x_{n-1}, 0). \quad (21)$$

Denoting by $t = (n - 1)/N$ the extensivity share already achieved, one step takes $\Delta t = 1/N$, and the above recursion can be Taylor-expanded for a small $y/N = y\Delta t$:

$$x(t + \Delta t) - x(t) = y \Delta t \left. \frac{\partial}{\partial y} h(x(t), y) \right|_{y=0^+} + \mathcal{O}(\Delta t^2). \quad (22)$$

In the large N ($\Delta t \rightarrow 0$) limit this becomes equivalent to a differential equation similar to a renormalization flow equation:

$$\frac{dx}{dt} = y h'_2(x, 0^+). \quad (23)$$

In this expression $h'_2(x, 0^+)$ denotes the partial derivative of the rule $h(x, y)$ with respect to its second argument taken when this value approaches zero from above. Note

that the uniformity of subdivisions to y/N is not necessary; all infinitesimal divisions summing up to y by $t = 1$ lead to the same differential flow equation.

The solution of eq.(23),

$$L(x) = \int_0^x \frac{dz}{h_2(z, 0^+)} = y t, \quad (24)$$

defines the additive mapping of x , i.e. the formal logarithm $L(x)$. By the help of this the following asymptotic composition rule arises:

$$x_{12} := \varphi(x_1, x_2) = L^{-1}(L(x_1) + L(x_2)); \quad (25)$$

it is already associative and commutative. Commutativity is trivial and associativity is also easily proved:

$$\begin{aligned} \varphi(\varphi(x_1, x_2), x_3) &= L^{-1}(L(\varphi(x_1, x_2)) + L(x_3)) \\ &= L^{-1}(L(x_1) + L(x_2) + L(x_3)) \\ &= L^{-1}(L(x_1) + L(\varphi(x_2, x_3))) \\ &= \varphi(x_1, \varphi(x_2, x_3)). \end{aligned} \quad (26)$$

It is interesting to check that *all associative rules are mapped to themselves in the above limit*. Given an associative composition rule, $h(x, y)$, it possesses a formal logarithm, $X(x)$, which is additive:

$$X(h(x, y)) = X(x) + X(y). \quad (27)$$

Now taking the derivative of this equality with respect to the second argument we obtain

$$X'(h) \partial h / \partial y = X'(y) \quad (28)$$

which taken at $y = 0$ becomes

$$h'_2(x, 0^+) = \frac{X'(0)}{X'(h(x, 0))}. \quad (29)$$

Due to the property $h(x, 0) = x$ (equivalently $X(0) = 0$) the formal logarithm of the asymptotic composition rule is given by

$$L(x) = \int_0^x \frac{X'(z)}{X'(0)} dz = \frac{X(x)}{X'(0)}; \quad (30)$$

it is proportional to the formal logarithm of the starting rule. Therefore the asymptotic rule is exactly the same as we begun with: $\varphi(x, y) = h(x, y)$. The freedom in a factor of the formal logarithm is used to set $X'(0) = 1$. This way any associative composition rule describes a limiting rule of a class of non-associative rules.

2.2 Deformed logarithms and deformed exponentials

The stationary distribution eq.(16) in the large- N limit contains the formal logarithm, $L(x)$. In fact the composed function, $e_a = \exp \circ L$ is the one, which is frequently called

a '*deformed exponential*' in the literature. Its inverse, $\ln_a = L^{-1} \circ \ln$ is then the corresponding '*deformed logarithm*'. These functions are inverse to each other. Further properties of the traditional exponential and logarithm functions are, however, not automatically inherited. In particular reciprocals and negatives follow different rules as we are used to.

In the particular case of scaling formal logarithms,

$$L_a(x) = \frac{1}{a} L_1(ax), \quad (31)$$

several interesting identities hold, among others the followings:

$$\begin{aligned} L_0(x) &= x, \\ L_a^{-1}(x) &= \frac{1}{a} L_1^{-1}(ax), \\ \ln_a(1/x) &= -\ln_{-a}(x), \\ 1/e_a(x) &= e_{-a}(-x) \end{aligned} \quad (32)$$

Since $a = q - 1$, the $a^* = -a$ duality corresponds to the $q^* = 2 - q$ Tsallis-duality. This can be important for the particle-hole relation for fermions:

$$1 - \frac{1}{e_a(-x) + 1} = \frac{1}{e_{-a}(x) + 1}. \quad (33)$$

Let us now list some important particular rules and their asymptotic pendants considered in applications of non-extensive statistics to physical systems.

The trivial (and classical) addition is the simplest composition rule: $h(x, y) = x + y$. In this case $h'_2(x, 0^+) = 1$ and one obtains

$$L(x) = \int_0^x dz = x. \quad (34)$$

The original Gibbs exponentials, $e^{-\beta E}/Z$, result as stationary distributions from any Monte Carlo type algorithm using the additive composition rule. The asymptotic rule is also the addition $\varphi(x, y) = x + y$.

Another rule leading to the so-called q -exponential distribution [13] is given by $h(x, y) = x + y + axy$ with the parameter a proportional to $q - 1$ occurring in the Pareto-Tsallis distribution. Now one obtains $h'_2(x, 0^+) = 1 + ax$ and

$$L(x) = \int_0^x \frac{dz}{1 + az} = \frac{1}{a} \ln(1 + ax). \quad (35)$$

This formal logarithm leads to a stationary distribution with power-law tail as the function composition $\exp \circ L$ on the power $-\beta$:

$$f(E) = \frac{1}{Z} e^{-\frac{\beta}{a} \ln(1 + aE)} = \frac{1}{Z} (1 + aE)^{-\beta/a}. \quad (36)$$

On the other hand, assuming such a non-additive composition rule for the generalized entropy, a special formula

can be constructed as the expectation value of the inverse of this function, of the deformed logarithm, $L^{-1} \circ \ln$. One obtains

$$S = \int f \frac{e^{-a \ln(f)} - 1}{a} = \frac{1}{a} \int (f^{1-a} - f). \quad (37)$$

The asymptotic composition rule again coincides with the original one: $\varphi(x, y) = x + y + axy$. We note here that the formal logarithm of the integrated expression is the (additive) Rényi entropy:

$$L(S) = \frac{1}{1-q} \ln \int f^q, \quad (38)$$

with $a = 1 - q$ and $\int f = 1$.

A further rule has been suggested by Kaniadakis[14], based on the sinh function. The formal logarithm is given as

$$L(x) = \frac{1}{\kappa} \text{Arsh}(\kappa x), \quad (39)$$

and its inverse becomes $L^{-1}(t) = \sinh(\kappa t)/\kappa$. The stationary distribution, composed by $\exp \circ L$, is

$$f_{\text{eq}}(p) = \frac{1}{Z} \left(\kappa p + \sqrt{1 + \kappa^2 p^2} \right)^{-\beta/\kappa}. \quad (40)$$

For large arguments it gives a power-law in the momentum p and hence also in the relativistic energy. The corresponding entropy formula is the average of $L^{-1} \circ \ln$ over the allowed phase space:

$$S_K = - \int \frac{f}{\kappa} \sinh(\kappa \ln f) = \int \frac{f^{1-\kappa} - f^{1+\kappa}}{2\kappa}. \quad (41)$$

The composition formula can be reduced to

$$h(x, y) = x\sqrt{1 + \kappa^2 y^2} + y\sqrt{1 + \kappa^2 x^2}. \quad (42)$$

For low arguments it is additive, $h(x, y) \approx x + y$, for high ones it is multiplicative, $h(x, y) \approx 2\kappa xy$. It has been motivated by the relativistic kinematics of massive particles. Interpreting the parameter as $\kappa = 1/mc$, one deals with $\kappa p = \sinh \eta$, so the formal logarithm becomes proportional to the rapidity, $L(p) = m\eta$. This implies a stationary distribution like $\exp(-\beta m\eta)$, which has not yet ever been observed in particle spectra stemming from relativistic heavy ion collisions. For such a purpose it is tempting to consider some further scenarios based on other quantities than suggested above (see next section).

The rule leading to a stretched exponential stationary distribution, often considered in problems related to anomalous diffusion and Levy-flights, is given by $h(x, y) = (x^b + y^b)^{1/b}$. Here the partial derivative is evaluated at a small positive argument, $\epsilon = y/2N$. One obtains $h'_2(x, \epsilon) = c(\epsilon)x^{1-b}$ with a factor depending on ϵ and for given values of b diverging in the $\epsilon = 0$ limit. However, this can be accommodated by our procedure; we obtain the formal logarithm $L(x) = c(\epsilon)x^b/b$, and therefore the asymptotic rule $\varphi(x, y) = (x^b + y^b)^{1/b}$. Again, constant factors in the

formal logarithm can be eliminated without loss of any information.

Now let us investigate a non-associative rule; its asymptotic limit cannot be itself. We regard a linear combination of arithmetic and harmonic means:

$$h(x, y) = x + y + a \frac{xy}{x + y} \quad (43)$$

The rescaling flow derivative is given by $h'_2(x, 0^+) = 1 + a$ and $-$ being a constant $-$ it leads to $L(x) = x/(1 + a)$ and with that to the *addition* as the asymptotic rule: $\varphi(x, y) = x + y$.

As an interesting rule we discuss the relativistic formula for collinear velocity composition,

$$h(x, y) = \frac{x + y}{1 + xy/c^2}. \quad (44)$$

This rule is associative, and it also preserves its form in the thermodynamic limit. The fiducial derivative is given by $h'_2(x, 0^+) = 1 - x^2/c^2$ and the formal logarithm, $L(x) = c \text{atanh}(x/c)$ turns out to be the rapidity. The asymptotic composition rule recovers the original one.

There are also general types of composition rules, which mutate into a simpler asymptotic form. For our discussion particularly important are rules of the form

$$h(x, y) = x + y + G(xy) \quad (45)$$

with a general function $G(z)$, restricted by the property $G(0) = 0$ only. In this case $h'_2(x, 0) = 1 + G'(0)x$ asymptotically leads to a Tsallis-Pareto distribution with the parameter $q - 1 = G'(0)$.

2.3 Extreme relativistic kinematics

In this section we review a particular type of pair interaction, which can be expressed as a function of the kinetic energies of the individual particles. The relation to relativistic kinematics is established by the fact, that we consider such dependence through the Lorentz-invariant relative four-momentum square variable:

$$E_{12} = E_1 + E_2 + U(Q^2). \quad (46)$$

We study whether relativistic speeds alone can cause "non-extensivity", i.e. a power-law tailed kinetic energy distribution. The relativistic formula for Q^2 is given by:

$$Q^2 = (\mathbf{p}_1 - \mathbf{p}_2)^2 - (E_1 - E_2)^2 \quad (47)$$

with \mathbf{p}_i, E_i being relativistic momenta and full energies of interacting bodies. Expressed by the energies and the angle Θ between the two momenta this becomes a linear expression of $\cos \Theta$:

$$Q^2 = 2(E_1 E_2 - p_1 p_2 \cos \Theta) - (m_1^2 + m_2^2) \quad (48)$$

with $p_i = \sqrt{E_i^2 - m_i^2}$ for $i = 1, 2$. Here we use relativistic units ($c = 1$) and assume the masses m_1 and m_2 , respectively, for the interacting partners. It is useful to note that writing eq.(48) as $Q^2 = 2(A - B \cos \Theta)$ we have

$$A \pm B = E_1 E_2 - \frac{1}{2}(m_1^2 + m_2^2) \pm p_1 p_2. \quad (49)$$

For the sake of simplification we average over the relative directions of the respective momenta and obtain

$$\begin{aligned}\langle U(Q^2) \rangle &= \frac{1}{2} \int_0^\pi U(2A - 2B \cos \Theta) \sin \Theta d\Theta \\ &= \frac{F(2A + 2B) - F(2A - 2B)}{4B},\end{aligned}\quad (50)$$

with $U(w) = dF/dw$. It is easy to derive by the substitution $w = 2(A - B \cos \Theta)$. The rule for the kinetic energy, $K_i = E_i - m_i$, composition is given by

$$K_{12} = K_1 + K_2 + \frac{F(2A + 2B) - F(2A - 2B)}{4B} \quad (51)$$

The quantities A and B^2 can be expressed by the respective kinetic energies and masses:

$$\begin{aligned}A &= K_1 K_2 + (m_2 K_1 + m_1 K_2) - \frac{1}{2}(m_1 - m_2)^2, \\ B^2 &= K_1 K_2 (K_1 + 2m_1)(K_2 + 2m_2).\end{aligned}\quad (52)$$

One observes that the product of kinetic energies occurs due to kinematic reasons.

Taylor expanding the integral of the unknown function $U(w)$ around $w = 2A$ and ensuring the $h(x, 0) = x$, as well as the $h(0, y) = y$ property, we obtain the following composition rule for the relativistic kinetic energies:

$$\begin{aligned}h(x, y) &= x + y - U(2m_2 x + m_{12}) - U(2m_1 y + m_{12}) \\ &\quad + U(m_{12}) + \sum_{j=0}^{\infty} U^{(2j)}(2A) \frac{(4B^2)^j}{(2j+1)!}\end{aligned}\quad (53)$$

with $m_{12} = -(m_1 - m_2)^2$, $A = xy + (m_2 x + m_1 y) + m_{12}/2$ and $4B^2 = 4xy(x + 2m_1)(y + 2m_2)$. For unequal masses, $m_1 \neq m_2$ this composition rule is not symmetric. Since

$$\begin{aligned}\frac{\partial A}{\partial y}(x, 0) &= m_1 + x, \\ \frac{\partial B^2}{\partial y}(x, 0) &= 2m_2 x(x + 2m_1),\end{aligned}\quad (54)$$

the derivative leading to the formal logarithm of the asymptotic rule becomes an expression with a finite number of terms

$$\begin{aligned}h'_2(x, 0) &= 1 - 2m_1 U'(m_{12}) + 2(m_1 + x) U'(z) \\ &\quad + \frac{4}{3} m_2 x (2m_1 + x) U''(z),\end{aligned}\quad (55)$$

with $z = 2A(x, 0) = 2m_2 x + m_{12}$. In all traditional approaches the interaction energy U is independent of Q^2 . In such cases $h'_2(x, 0) = 1$ and the simple addition is the asymptotic composition rule. Therefore the stationary energy distribution is of Boltzmann-Gibbs type. For Q^2 dependent interactions on the other hand it is important to consider the extreme relativistic kinematics. In this case the replacement $m_1 = m_2 = 0$ leads directly to

$$h'_2(x, 0) = 1 + 2x U'(0). \quad (56)$$

As discussed in the previous subsection this generates a Tsallis-Pareto distribution in the relativistic kinetic energy. This result includes for $U' = 0$ the traditional momentum independent interaction case leading to the addition as asymptotic rule for non-relativistic kinetic energies, and hence to the Boltzmann-Gibbs distribution. We note that in the relativistic kinematics the linear assumption, $U' = \alpha = \text{const.}$ also leads to a Tsallis-Pareto distribution due to $h'_2(x, 0) = 1 + 2\alpha x$.

2.4 Generalized entropies to each composition rule

There are two possible approaches in constructing a generalized entropy formula: i) either to use a non-additive entropy for independent events with factorizing probability, or ii) to search for an additive entropy while the common probability is not factorizing in the individual probabilities. In both cases the entropy density function, $\sigma(p)$ to a probability p can be obtained from the composition rule $h(x, y)$.

First we consider a non-additive entropy formula for factorizing probabilities, i.e.

$$\sum_{i,j} w_{ij} \sigma(w_{ij}) = h \left(\sum_i p_i \sigma(p_i), \sum_j q_j \sigma(q_j) \right) \quad (57)$$

with $w_{ij} = p_i q_j$. We would like to construct the function $\sigma(p)$ by knowing $h(x, y)$. Let us inspect the equipartition case, $p_i = 1/N_1$, $q_j = 1/N_2$. In this case $w_{ij} = 1/(N_1 N_2)$. Eq.(57) leads to

$$\sigma(ab) = h(\sigma(a), \sigma(b)) \quad (58)$$

with $a = 1/N_1$ and $b = 1/N_2$. This requires the same composition rule for micro- and macro-entropy:

$$\sum_{i,j} p_i q_j h(\sigma(p_i), \sigma(q_j)) = h \left(\sum_i p_i \sigma(p_i), \sum_j q_j \sigma(q_j) \right). \quad (59)$$

This h -extensivity can so far only be satisfied by the Tsallis rule $h(x, y) = x + y + axy$. On the other hand if $q_0 = 1$ and all other $q_j = 0$ for $j \neq 0$, we obtain two constraints:

$$\begin{aligned}\sigma(p_i) &= h(\sigma(p_i), \sigma(1)) \\ \sigma(0) &= h(\sigma(p_i), \sigma(0))\end{aligned}\quad (60)$$

from which it follows $h(x, 0) = x$ with $\sigma(1) = 0$ (the unexpectedness of a sure event is zero) and $\sigma(0) = \infty$, too.

Based on the properties of the known $h(x, y)$, in the thermodynamical limit it is associative and hence possess a formal logarithm, $L(x)$. Therefore

$$L(\sigma(ab)) = L(\sigma(a)) + L(\sigma(b)), \quad (61)$$

whose general solution is given by $L \circ \sigma = \beta \ln$. According to the tradition $\beta = -1$ in units of the Boltzmann constant, $k_B = 1$, and therefore the entropy density function is expressed by the deformed logarithm:

$$\sigma(p) = L^{-1}(-\ln p) = \ln_a \left(\frac{1}{p} \right). \quad (62)$$

It is possible to ask another question: if the construction rule for the common probability is not the simple product, but it is known, what should the entropy density function be in order to lead to the addition rule for the total entropy. So given the formula

$$w_{ij} = e^{h(\ln p_i, \ln q_j)} \quad (63)$$

how to construct $\sigma(p)$ such that

$$\sum_{ij} w_{ij} \sigma(w_{ij}) = \sum_i p_i \sigma(p_i) + \sum_j q_j \sigma(q_j) \quad (64)$$

is fulfilled. We have

$$c = w_{ij} = e^{h(\ln a, \ln b)} \quad (65)$$

as the known composite probability and seek for the entropy density function, $\sigma(a)$ satisfying

$$c \sigma(c) = ab (\sigma(a) + \sigma(b)). \quad (66)$$

We solve this functional equation by deriving with respect to b and take the result at $b = 1$. Since

$$\frac{\partial c}{\partial b} = e^{h(\ln a, \ln b)} h'_2(\ln a, \ln b) \frac{1}{b} \quad (67)$$

we arrive at

$$a h'_2(\ln a, 0) (\sigma(a) + a \sigma'(a)) = a \sigma(a) + a \sigma(1) + a \sigma'(1). \quad (68)$$

Using now that $\sigma(1) = 0$ and $h'_2(x, 0) = 1/L'(x)$ with the formal logarithm L associated to the composition rule, h , we obtain – using the variable $x = \ln a$ –

$$\frac{d\sigma}{dx} + \sigma = L'(x) (\sigma + \beta), \quad (69)$$

with $\beta = \sigma'(1)$ constant. The final solution is expressed by the formal logarithm of the asymptotic rule as

$$\sigma(a) = \beta e^{L(\ln a) - \ln a} \int_0^{\ln a} L'(u) e^{u - L(u)} du. \quad (70)$$

It is interesting to note, that using L_a^{-1} of the deformed logarithm as the function L belonging to the product composition rule (65), one assumes $\ln_a(w_{ij}) = \ln_a(p_i) + \ln_a(q_j)$ and arrives at

$$p \sigma(p) = -e^{\ln_a p} \int_0^{\ln_a p} e^{-u} e_a(u) du. \quad (71)$$

3 Non-extensive Boltzmann equation

It is a false belief that only the exponential distribution can be the stationary solution to the Boltzmann equation: this statement is true only i) if the two-particle distributions factorize, ii) the two-particle energies are additively composed from the single-particle energies ($E_{12} =$

$h(E_1, E_2) = E_1 + E_2$) and iii) the collision rate is multilinear in the two-particle (and two-hole) densities. A generalization of the original Boltzmann equation has been pioneered by Kaniadakis[15] investigating nonlinear density dependence of the collision rates. An " H_q " theorem for the particular Tsallis form of the collision rate has been derived by Lima, Silva and Plastino[16].

A possible generalization of the Boltzmann equation uses an altered form of the 'Stosszahlansatz' and allows for an evolution equation of a function of the original phase space occupation factor, $F(f)$:

$$DF(f_1) = \int_{234} w_{1234} (G_{34} - G_{12}) \quad (72)$$

with

$$DF = \frac{p^\mu}{p^0} \partial_\mu F \quad (73)$$

total (Vlasov-) derivative, with a 1234-symmetric collision rate including Dirac-delta distributions for momentum and energy composition rules in two-to-two collisions (which also may be of generalized type by using corresponding formal logarithms), and finally the generalized product for the two-particle density factor,

$$G_{12} = e_a(\ln_a(f_1) + \ln_a(f_2)) \quad (74)$$

using the deformed exponential and logarithm functions. Based on this, a particular expression for the entropy current density can be defined:

$$S^\mu = - \int \frac{p^\mu}{p^0} \sigma(F(f)). \quad (75)$$

The entropy density form, $\sigma(F)$ always can be constructed in a way, that the second theorem of thermodynamics is fulfilled. The local source for the entropy is namely given by

$$\partial_\mu S^\mu = - \int_1 \sigma'(F(f_1)) DF(f_1). \quad (76)$$

Utilizing the generalized Boltzmann equation (72) and exchanging the index 1 with 2, 3 and 4 while w_{1234} stays invariant and obviously $G_{ij} = G_{ji}$, one arrives at

$$\partial_\mu S^\mu = \frac{1}{4} \int_{1234} w_{1234} (\sigma'_1 + \sigma'_2 - \sigma'_3 - \sigma'_4) (G_{12} - G_{34}) \quad (77)$$

with $\sigma'_i = \sigma'(F(f_i))$ for $i = 1, 2, 3, 4$. This quantity is always non-negative, i.e.

$$(\Phi(G_{12}) - \Phi(G_{34})) (G_{12} - G_{34}) \geq 0, \quad (78)$$

if and only if

$$\Phi(G_{12}) = \sigma'(F(f_1)) + \sigma'(F(f_2)) \quad (79)$$

is a monotonic rising function. Inspecting the generalized Stosszahlansatz eq.(74) one finds that this splitting to the

sum of respective functions of f_1 and f_2 is only possible, if $\Phi(t) \propto \ln_a(t)$. Therefore we conclude that

$$\sigma'(F(f)) = \alpha \ln_a(f) + \beta \quad (80)$$

with $\alpha \geq 0$ and β undetermined constants. (This derivation followed the spirit of Ref.[15].)

The generalized entropy density as a function of the one-particle phase space occupation density is hence given by

$$\sigma(f) = \int F'(f) (\alpha \ln_a(f) + \beta) df. \quad (81)$$

The traditional Boltzmann formula arises for $F(f) = f$ and $\ln_a(f) = \ln(f)$ (i.e. $a = 0$). Lavagno et.al. [17] considered $F(f) = f^q$ and $\ln_a(f) = (f^{q-1} - 1)/(q - 1)$ (i.e. $a = (q - 1)$ and Tsallis composition rule for $\ln f$). In the case of $h(x, y) = x + y + axy$ one considers $\ln_a(x) = (x^a - 1)/a$, $e_a(t) = (1 + at)^{1/a}$ and $G_{12} = (f_1^a + f_2^a - 1)^{1/a}$. For a small a parameter it is $G_{12} \approx f_1 f_2 (1 - a \ln(f_1) \ln(f_2) + \dots)$.

We note that the detailed balance distribution is given by the condition $G_{12} = G_{34}$, while the corresponding energy composition rule applies $L(E_1) + L(E_2) = L(E_3) + L(E_4)$. This is possible only if $\ln_a(f_i) = -(L(E_i) - \mu)/T$, so

$$f^{\text{eq}}(E) = e_a \left(\frac{\mu - L(E)}{T} \right). \quad (82)$$

The parameters T and μ are arbitrary constants for being a stationary solution of the generalized Boltzmann equation, but they can be related to the total energy and particle number in a given application.

3.1 Deformed energy composition rules in parton cascade

Our first numerical approach [9] was restricted to the use of abstract composition rules in the energy balance part: we equated the energy of the reacting parts before and after the collision via an abstract energy composition rule

$$h(E_1, E_2) = h(E_3, E_4). \quad (83)$$

Although this rule cannot be specified without further knowledge, according to our results presented in the previous section, in the thermodynamical limit an asymptotic rule can be considered, with a formal logarithm. The parton cascade simulation based on a Boltzmann equation is hence modified by considering

$$L(E_1) + L(E_2) = L(E_3) + L(E_4). \quad (84)$$

At the same time we applied $F(f) = f$ and $a = 0$. Applying such a general energy composition rule considered in the thermodynamical limit, the rate of change of the one-particle distribution is given by

$$\dot{f}_1 = \int_{234} w_{1234} [f_3 f_4 - f_1 f_2]. \quad (85)$$

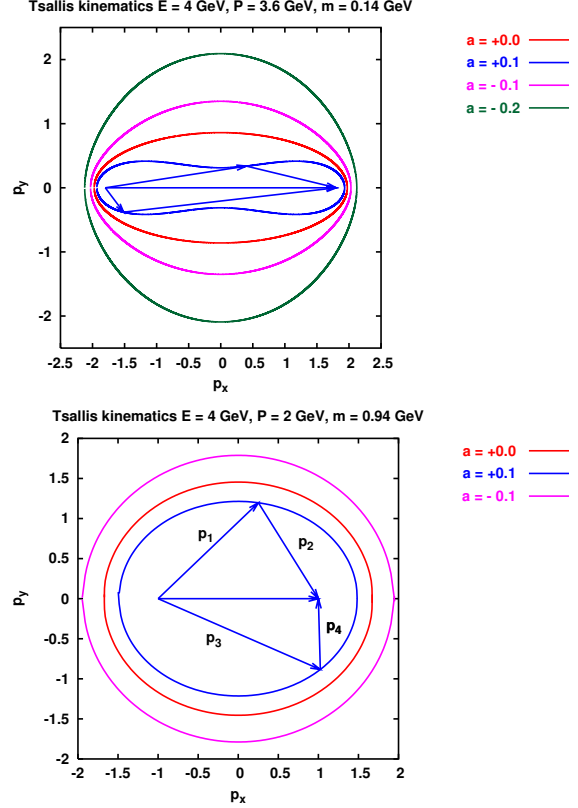


Fig. 2. Momentum vectors for pions (upper) and protons (lower) at $E = 4$ GeV pair energy according to eq.(86) with the rule $h(x, y) = x + y + axy$ for the energies.

with the symmetric transition probability w_{1234} including the constraint

$$\Delta = \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta(h(E_1, E_2) - h(E_3, E_4)). \quad (86)$$

In the figure 2 the possible pairs of momentum vectors are shown for the $h(x, y) = x + y + axy$ energy composition rule for pions ($m = 0.14$ GeV mass) and protons ($m = 0.94$ GeV mass), respectively. The two-dimensional cuts for the endpoints of the respective vectors form an ellipsoid in the traditional $a = 0$ case, while this surface is deformed for nonzero extensivity parameters, as seen in the figure.

In a stationary state the $f(E_i)$ distributions depend on the phase space points through the energy variables only (this is to be checked on experimentally observed hadron transverse momentum spectra at mid-rapidity by the so called m_T -scaling) and the detailed balance principle requires

$$f(E_1) f(E_2) = f(E_3) f(E_4). \quad (87)$$

With the generalized constraint (86) this relation is satisfied by

$$f(E) = f(0) \exp(-L(E)/T). \quad (88)$$

For the Tsallis-type energy addition rule[18,19], one obtains cut power-law stationary distribution,

$$f(E) = f(0) (1 + bE)^{-1/bT}. \quad (89)$$

Connecting this to the Tsallis parametrization one uses $q = 1 + bT$. Since the energy addition rule conserves in a two by two collision the quantity $h(E_1, E_2)$, the new energies after such an event lie on the $h(E_3, E_4) = \text{constant}$ surface. Due to the additivity of the formal logarithm of the single particle kinetic energies, $L(E_i)$, the total sum $L_{tot} = \sum_i L(E_i)$, is a conserved quantity. This rule was applied in numerical simulations [9, 22]. During the numerical searches for stationary distributions only the tacit assumption of constant transition probability rates has been applied; the evolution results are obtained in terms of the number of pairwise momentum exchange events, not in terms of real time.

3.2 Random momenta

Parton cascade simulations usually consider pairwise collisions with energy and momentum conservation inside the two-particle system. The pairs to collide are chosen randomly from an ensemble of particles and the new momenta are generated randomly according to the above constraints. This way the probability is uniform in the two-particle phase space, provided the conditions for momentum and energy sums (in our more general case for the energy composition) are satisfied:

$$d^2w = w_0 \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) \delta(h(E_1, E_2) - H) d^3p_1 d^3p_2. \quad (90)$$

The constant w_0 is fixed by the normalization of the integral of this probability density to one (or to the actual collision rate in real-time simulations). Since there are six degrees of freedom and four constraints, two free quantities have to be chosen randomly. It is, however, a delicate procedure to ensure the random uniformity in the two-particle phase space for a general energy composition rule.

It is customary to introduce the sum and difference of the momentum vectors by

$$\mathbf{p}_{1,2} = \frac{1}{2} \mathbf{P} \pm \mathbf{q}. \quad (91)$$

Using this notation the momentum sum constraint can be integrated out trivially and - since the Jacobean of the transformation (91) is one - we arrive at

$$d^2w = w_0 \delta(h(E_1, E_2) - H) d^3q. \quad (92)$$

For the addition rule, $h(E_1, E_2) = E_1 + E_2$, it is enough to obtain the direction of the vector \mathbf{q} accordingly while its magnitude is constrained by the energy sum. It is a straightforward task to do it in the center of mass system, where the momentum sum vector, \mathbf{P} , vanishes: the direction of the difference vector \mathbf{q} in this system is uniform on a spherical surface. A Lorentz-transformation into this system, a random azimuthal angle and a random cosine, and finally a back transformation provide the new momenta after a collision.

Since we are dealing with a constraint more general in the energy variables, first we transform the problem of

randomly choosing the difference vector \mathbf{q} into a problem of choosing proper energies after the collision. The energies are expressed by the free dispersion relations

$$E_{1,2}^2 - m_{1,2}^2 = \frac{1}{4} P^2 + q^2 \pm Pq \cos \theta, \quad (93)$$

where P and q denote the lengths of the corresponding vectors and θ the angle between them. From this two equations one easily derives the following energy differentials:

$$\begin{aligned} 2E_1 dE_1 &= 2q dq + P \cos \theta d\theta - Pq \sin \theta d\theta, \\ 2E_2 dE_2 &= 2q dq - P \cos \theta d\theta + Pq \sin \theta d\theta. \end{aligned} \quad (94)$$

The phase space volume element can be expressed easily by using the wedge product form:

$$d^3q = dq \wedge q \sin \theta d\theta \wedge q d\phi \quad (95)$$

which upon using eq.(94) can be written as

$$d^3q = \frac{E_1 E_2}{P} dE_1 \wedge dE_2 \wedge d\phi. \quad (96)$$

Now using the energy composition constraint we arrive at a probability density which is not uniform in the energy:

$$d^2w = w_0 \delta(h(E_1, E_2) - H) \frac{E_1 E_2}{P} dE_1 dE_2 d\phi. \quad (97)$$

One uses the constraint to eliminate say E_2 from the above formula and considers

$$d^2w = w_0 \frac{E_1 E_2}{P h'_2(E_1, E_2)} dE_1 d\phi. \quad (98)$$

In the general case the differential probability density, dw/dE_1 , is a complicated function of the energy. Its integral, $w(E_1)$ has to be uniformly distributed.

In the case of a Tsallis composition rule one obtains $E_2 = (H - E_1)/(1 + aE_1)$ and we arrive at

$$d^2w = \frac{w_0}{P} \frac{E_1 (H - E_1)}{(1 + aE_1)^2} dE_1 d\phi. \quad (99)$$

This expression can be integrated giving

$$d^2w = \frac{1}{2\pi} d\rho d\phi, \quad (100)$$

with

$$\rho(E_1, a) = \frac{(2 + aH) \ln(1 + aE_1) - \frac{aE_1}{1 + aE_1} (2 + aH + aE_1)}{(2 + aH) \ln(1 + aH) - 2aH} \quad (101)$$

when properly normalized. The only problem is that $\rho(E_1)$ cannot be inverted analytically. Even in the traditional case with $a = 0$, the inversion requires the solution of a third order equation:

$$\rho(E, a = 0) = 3(E/H)^2 - 2(E/H)^3 \quad (102)$$

is distributed uniformly between zero and one. It means that E is between zero and H , the total composed energy.

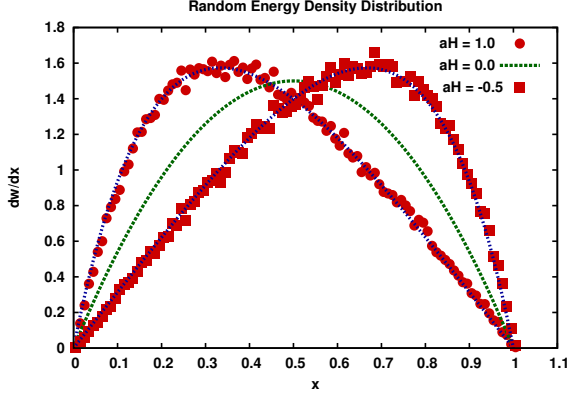


Fig. 3. The scaled differential probability density, dw/dx is shown as a function of the random energy share of one of the collided particles $x = E_1/H$ (cf. eq.(99)) for different values of aH . Full circles and boxes denote the random energy deviates obtained numerically using the rejection method.

After having E_1 and E_2 the momenta can be reconstructed with the help of a vector triad describing the direction of the momentum sum, \mathbf{P} . They are given as $n_i = P_i/P$ and $e_i = (-n_\perp, n_1 n_2 / n_\perp, n_1 n_3 / n_\perp)$ where the notation $n_\perp = \sqrt{n_2^2 + n_3^2}$ stands for the component perpendicular to the first axis. The third orthogonal unit vector is $f_i = (0, -n_3/n_\perp, n_2/n_\perp)$. The momentum difference vector is hence reconstructed as

$$\mathbf{q} = q_{||} \mathbf{n} + q_\perp (\cos \phi \mathbf{e} + \sin \phi \mathbf{f}) \quad (103)$$

with

$$\begin{aligned} q_{||} &= \frac{E_1^2 - E_2^2}{2P}, \\ q^2 &= \frac{E_1^2 + E_2^2}{2} - \frac{P^2}{4}, \\ q_\perp &= \sqrt{q^2 - q_{||}^2}. \end{aligned} \quad (104)$$

3.3 Parton cascade simulation

First we show some snapshots of the colliding partons in the $p_x - p_y$ phase space cut at different stages of the evolution marked by the average number of collisions per particle, t (cf. Fig.4). At the beginning $t = 0$ we prepared two distributions at a given energy per particle and then Lorentz boosted each with $y_B = 2$ units of rapidity in opposite ways in the p_x -direction. The dark dots represent particle momenta stemming from the respectively boosted original sets. The evolution towards a zero centered and isotropic distribution of momenta signals already that thermal equilibration happens.

Fig.5 presents results of a simple test particle simulation with the rule $h(x, y) = x + y + axy$ with $a = 0$ (left) and $a = 2$ (right), respectively. We started with a uniform energy-shell distribution between zero and $E_0 = 1$ with

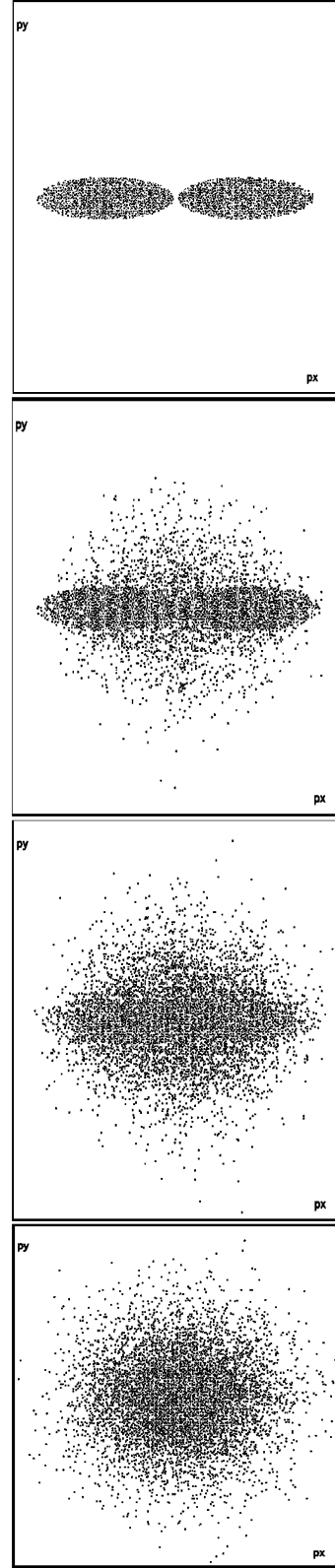


Fig. 4. Snapshots of phase space cuts in the $p_x - p_y$ plane for colliding partons with the deformed energy composition rule $h(x, y) = x + y - 0.2 xy$ at $t = 0, 0.3, 1$, and 3 (from top to bottom).

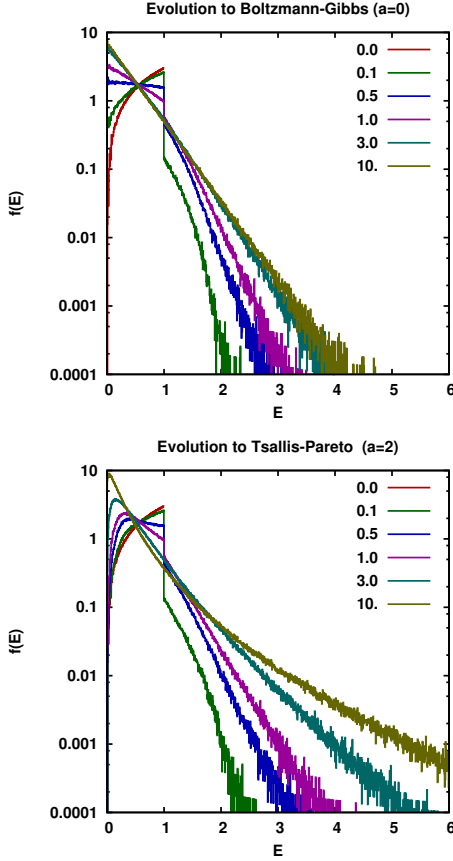


Fig. 5. Evolution of single particle energy kinetic energy distributions for massless particles towards the Boltzmann-Gibbs distribution for $h(E_1, E_2) = E_1 + E_2$ (left part) and towards the Tsallis-Pareto one (right part) for $h(E_1, E_2) = E_1 + E_2 + 2E_1E_2$. The curves are normalized to the same integral $\int E^2 f(E) dE$.

a fixed number of particles $N = 10^6$. The one-particle energy distribution evolves towards the well-known exponential curve for $a = 0$, shown in the left part of Fig. 5. These snapshots were taken initially and after 0.1, 0.5, 1, 3 and 10 two-body collisions per particle. Using the prescription with $a = 2$, the energy distribution approaches a Tsallis-Pareto distribution.

It is in order to make some remark on the energy conservation. For $h(x, y) = x + y$ we simulate a closed system with elastic collisions: The sum, $E_{\text{tot}} = \sum_{i=1}^N E_i$, does not change in any of the binary collisions. This is different by using a non-extensive formula for $h(x, y)$. With a constant positive (negative) a , the bare energy sum is decreasing (increasing) while approaching the stationary distribution, while the sum of the formal logarithms of the energy remains constant. Open systems may gain or loose energy during their evolution towards a stationary state.

4 Non-extensive thermal equilibration

In order to investigate the equilibration of non-extensive systems we start with two subsystems, equilibrated separately. In order to prepare these systems the non-extensive Boltzmann equation can be solved numerically in a parton cascade simulation as described in the previous section. As an alternative way we use initial momentum distributions prepared by Monte Carlo rejection techniques in the form of eq. (88), with different energy per particle but a common parameter a for the one and the other half of the particles. Then random binary collisions between randomly chosen pairs of particles are evaluated. By doing so we apply the rules

$$X(E_1) + X(E_2) = X(E_3) + X(E_4), \quad (105)$$

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4. \quad (106)$$

In each step of the simulation we select two particles to collide. Then we find the value for the new momentum of the first particle (\mathbf{p}_3) satisfying the above constraints but otherwise random. Then applying eq. 106 we calculate the momentum of the second outgoing particle (\mathbf{p}_4). In these particular simulations we use the free dispersion relation for massless particles ($E_i(\mathbf{p}_i) = |\mathbf{p}_i|$), since we are interested in the extreme relativistic kinematics case. A typical simulation includes $10^6 - 10^7$ collisions among $10^5 - 10^6$ particles. After 3 – 5 collisions per particle on the average, the one-particle distribution approaches its stationary form sufficiently.

The following quantities are conserved during the simulation:

$$X(E_{\text{tot}}) = \sum_{i=1}^N X(E_i), \quad \mathbf{P} = \sum_{i=1}^N \mathbf{p}_i, \quad N = \sum_{i=1}^N 1. \quad (107)$$

We use the rule $h(x, y) = x + y + axy$ for the energy composition, here $a \sim (q-1)/T$ is the non-extensivity parameter. Our model reconstructs the traditional Boltzmann-Gibbs thermodynamics in the limit of $a = 0$.

As a preparation for the study of non-extensive thermal equilibration, we perform simulations on two large subsystems with particle numbers $N_1 = N/2$ and $N_2 = N/2$, total (quasi)energies $X(E_1)$ and $X(E_2)$ and non-extensivity parameter a . The unified system is taken as an initial state with $N = N_1 + N_2$ particles.

4.1 Simulation results

Our results show that the subsystems do equilibrate, they tend towards having a common stationary distribution.

We present examples with different initial conditions. We fix the particle numbers for each subsystem, $N_1 = N_2 = 100\,000$. The number of collisions in a typical simulation is $N_{\text{coll}} = 1\,000\,000$, so that $N_{\text{coll}}/(N_1 + N_2) = 5$ collisions happen per particle. This quantity we use as an evolution parameter instead of the real time. This way we do not have to know differential cross sections; from the

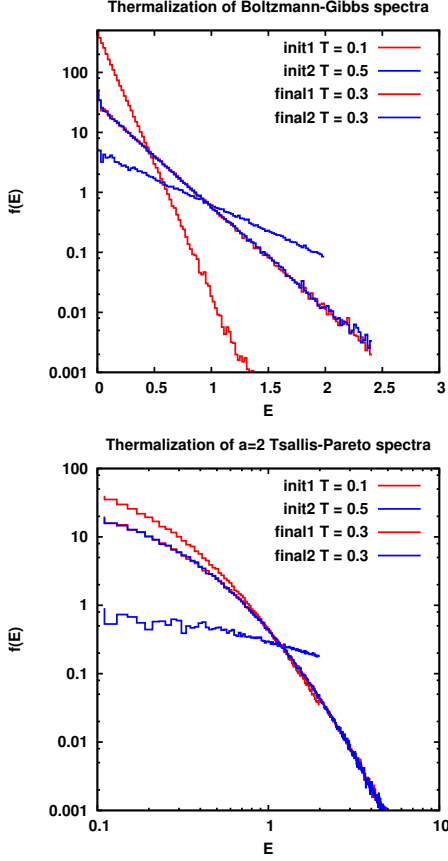


Fig. 6. Equilibration of two Boltzmann-Gibbs systems ($a = 0$, upper figure) plotted on a linear - logarithmic scale and equilibration of two Tsallis-type non-extensive systems ($a = 2$, lower figure) plotted on a double logarithmic scale. These are results for three-dimensional systems with 10 collisions per particle on the average. Each subsystem consists of 100.000 massless particles.

viewpoint of the fact of equilibration its rapidity does not matter.

In the figure 6 energy distribution curves are shown: the initial and the final ones and the ones after 5 collisions per particle, respectively. The upper part plots a Boltzmann system (simulation with $a = 0$) the lower one a Tsallis system with the energy composition rule using $a = 2$. In the upper half a logarithmic - linear plot is shown while in the lower half a double logarithmic plot. These choices are selected by the respective high energy asymptotics; exponential for a Boltzmann-Gibbs, while power-law for a Tsallis-Pareto distribution. It is hard to distinguish the energy distributions in the subsystems in the final state, the simulation curves are very close to each other. Therefore we conclude, that within numerical uncertainties a common stationary energy distribution is achieved.

4.2 Equilibration of large subsystems

Seeking for a canonical equilibrium state we have to maximize the total entropy given by a general composition rule, $S(E_1, E_2)$, at the same time satisfying a constraint which is in the general case also non-additive: $h(E_1, E_2)$ is constant. For the moment we neglect the dependence on further thermodynamical variables; usually the particle number N and the volume V is regarded to be proportional and extensive.

In the traditional case both the entropy and the energy are combined additively: $S(E_1, E_2) = S(E_1) + S(E_2)$ and $h(E_1, E_2) = E_1 + E_2$. In the general case by using corresponding formal logarithms the quantities $Y(S)$ and $X(E)$ have to be considered as additive. Since for associative rules the formal logarithm is strict monotonic, the maximum of the total entropy is achieved where $Y(S)$ has its extreme. The general canonical principle is therefore given by

$$Y(S) - \beta X(E) = \max. \quad (108)$$

The parameter β at this point is a Lagrange multiplier. Applying this for the equilibration of two large subsystems, and assuming that the entropy of each systems depends only on its own energy, one arrives at the equilibrium condition

$$\frac{Y'(S(E_1))}{X'(E_1)} S'(E_1) = \frac{Y'(S(E_2))}{X'(E_2)} S'(E_2) = \frac{1}{T}. \quad (109)$$

Comparing this with the general canonical form eq.(108) we obtain that $\beta = 1/T$, and T is an absolute temperature in the classical thermodynamical sense. Its relation to the entropy, however, has been generalized. In particular for an additive entropy, but non-additive energy composition rule, one arrives at $1/T = S'(E)/X'(E)$. The relation of this quantity to the logarithmic spectral slope, $1/T_{\text{slope}} = -d \ln f / dE = S'(E)$ leads to a practical tool for the analysis of particle spectra in experiments. For the Pareto-Tsallis distribution it is given by $T_{\text{slope}} = T/X'(E) = T(1 + aE) = T + (q - 1)E$. The naive effort to extract a temperature from energy spectra of particles, as it is a widespread usage in relativistic heavy ion studies, only works if $q = 1$, i.e. for spectra exponential in the particle energy. Otherwise an energy dependent slope, and a curved spectrum in the logarithmic plot has to be interpreted.

The inverse logarithmic slopes of single-particle kinetic energy spectra in the generalized case are functions of the energy:

$$T_{\text{slope}} = \frac{-1}{\frac{\partial}{\partial E} \ln f(E)}. \quad (110)$$

For the Tsallis-Pareto distribution they are linear functions, $T_{\text{slope}} = T + (q - 1)E$. Such slope parameters are plotted in Ref.[22] for the respective subsystems before and after equilibration (10 collisions per particle on the average). Within numerical uncertainties it is clear that common- a systems do equilibrate at a common temperature also in the $a \neq 0$ case.

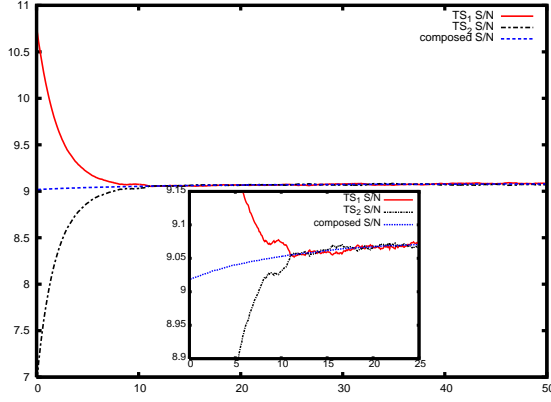


Fig. 7. The evolution of the Boltzmann entropy per particle during collisions with non-additive energy composition rules: the hotter body cools, the cooler body warms up, while the total entropy also increases. In the insertion a magnification of the curves is shown.

The (in our case Boltzmann) entropy also evolves due to the collisions. In Fig.7 the evolution of the entropy per particle is plotted for the hot and cool subsystems, and for the total system respectively. Since the composite system is combined from equal numbers of particles in each subsystem, the total entropy per particle starts with the arithmetic mean of the respective specific entropies. This value, however, rises somewhat, featuring a trend according to the second law of thermodynamics.

5 Power-law tailed hadron spectra, flow and quark coalescence

As an application of the above reviewed treatment of non-extensivity, in this section we demonstrate that hadronic transverse momentum spectra stemming from relativistic heavy ion collisions can be well described by cut power-law spectra in statistical models. In order to do so, one has to disentangle effects of a possible transverse flow on these spectra and then test whether the result complies with the thermal assumption; i.e. that the dependence on momenta is through a dependence on the kinetic energy, $E - \mu = E - m$ only. The $\mu = m$ assumption corresponds to a vanishing Fermi momentum for fermions, so this is the natural assumption at zero net baryon density. Therefore transverse momentum spectra at mid-rapidity are expected to follow such statistical model assumptions the best.

5.1 Spectral temperatures in relativistic heavy ion collisions

It has been long discussed, how a temperature can be conjectured from observations on particle spectra produced in relativistic heavy ion collisions. One intriguing way is

to look at the transverse momentum, p_T , spectra around mid-rapidity. The different identified hadrons, mostly pions, kaons, protons and antiprotons, have to demonstrate that their abundance in the momentum space depends on their kinetic energy; this phenomenon at zero rapidity is the so-called $m_T - m$ -scaling. The transverse mass is given as $m_T = \sqrt{m^2 + p_T^2}$, at strictly zero rapidity this is the total relativistic energy.

The analysis is made a little more involved by the fact that the source emitting the detected hadrons is flowing in all directions. The most prominent effects are due to a relativistic transverse flow with velocity v_T (and a corresponding Lorentz factor $\gamma_T = 1/\sqrt{1 - v_T^2}$ in units where $c = 1$). The relativistic energy of a particle in the frame of the emitting source cell is given by the Jüttner variable:

$$E = u_\mu p^\mu = \gamma_T m_T \cosh(y - \eta) - \gamma_T v_T p_T \cos(\varphi - \Phi). \quad (111)$$

Here the four-velocity of the source and the actual four-momentum of the particle are parametrized by rapidity and angle variables:

$$\begin{aligned} u_\mu &= (\gamma_T \cosh \eta, \gamma_T \sinh \eta, \gamma_T v_T \cos \Phi, \gamma_T v_T \sin \Phi), \\ p_\mu &= (m_T \cosh y, m_T \sinh y, p_T \cos \varphi, p_T \sin \varphi). \end{aligned} \quad (112)$$

We consider a thermal model for the particle spectra; then the yield is supposed to depend on the Jüttner variable E given by eq.(111). Assuming a general distribution $f(E) \sim \exp(-(X(E) - m)/T)$, which is monotonic decreasing, one finds its maximum at the minimum of E . This variable is minimal at the rapidity $y_{\min} = \eta$, and angle $\varphi_{\min} = \Phi$, giving

$$E_{\min} = \gamma_T m_T - \gamma_T v_T p_T. \quad (113)$$

This Lorentz-boosted transverse energy reaches its minimum at the transverse momentum value $p_{T,\min} = m \gamma_T v_T$, leading to $m_{T,\min} = m \gamma_T$ and $E_{\min} = m$. The expansion around this minimum in the p_T -distribution is an effective Gaussian:

$$e^{-(E-m)/T} \approx \exp\left(-\frac{(p_T - m \gamma_T v_T)^2}{2m \gamma_T T \gamma_T}\right). \quad (114)$$

In fact, according to experimental findings at RHIC the observed particle spectra have to be corrected for a transverse flow in order to reach m_T -scaling.

5.2 Non-extensivity in quark matter and in hadron matter

We conjecture that the power-law tails observed in hadronic spectra may stem from non-extensivity of the suddenly hadronizing quark matter. We look for a connection between quark and hadron spectra in the framework of the quark coalescence model. A coalescence of a quark and an antiquark into a meson produces a yield proportional to the quantity:

$$F(\mathbf{p}) = \int f(E(\mathbf{P}/2 + \mathbf{q})) f(E(\mathbf{P}/2 - \mathbf{q})) C(\mathbf{q}) d^3q. \quad (115)$$

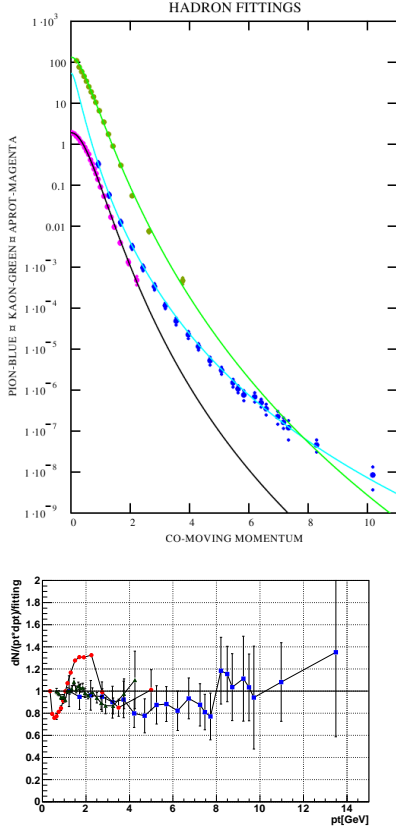


Fig. 8. General shape of p_T spectra for pions, kaons and antiprotons in relativistic heavy ion experiments (upper figure). A fit is done by using for $X(E)$ the Tsallis-Pareto form with parameters T and a , corresponding to a common temperature of $T(m_i) = 0.160$ MeV for the different particles. and a transverse flow velocity $v_T = 0.52$. In the lower part the ratio of the Tsallis fit to the experimental values can be inspected in a linear plot.

Here we integrate over the relative momentum of the quarks with a coalescence factor, $C(\mathbf{q})$, for which a simple model has been utilized [23]. For common momenta much larger than the relative one $|\mathbf{P}| \gg |\mathbf{q}|$ one obtains

$$F(\mathbf{P}) \approx f^2(E(\mathbf{P}/2)) \int C(\mathbf{q}) d^3q. \quad (116)$$

In particular light hadrons made from massless quarks follow the quark-scaling rule:

$$f_{\text{hadron}}(E) \propto f^n(E/n). \quad (117)$$

As a consequence particular properties of the non-extensive thermal model between quark and hadron matter also scale: $T_{\text{mesons}} = T_{\text{baryons}} = T_{\text{quarks}}$ for the temperature, while $q_{\text{mesons}} - 1 = (q_{\text{quarks}} - 1)/2$ for mesons and $q_{\text{baryons}} - 1 = (q_{\text{quarks}} - 1)/3$ for baryons. Since for a Tsallis-Pareto distribution the inverse logarithmic slope turns out to be

$$T_{\text{slope}} = T + (q - 1)(E - m), \quad (118)$$

the rise of these slopes reflect the non-extensivity parameters. In Fig.9 we show the test of the coalescence model prediction for the meson to baryon ratio.

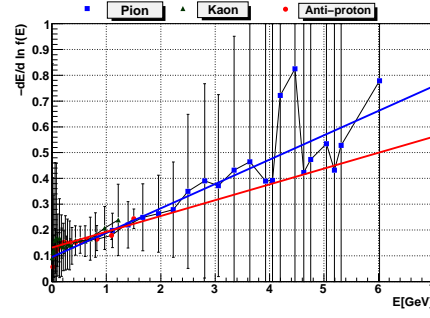


Fig. 9. Inverse logarithmic slopes, $T_{\text{slope}}(E) = -dE/d \ln f(E)$ extracted by numerical derivation from the experimental hadronic spectra (after subtracting a common flow effect). The full lines correspond to a common meson and baryon fit their steepness keeping the ratio 2:3 predicted by the quark coalescence picture.

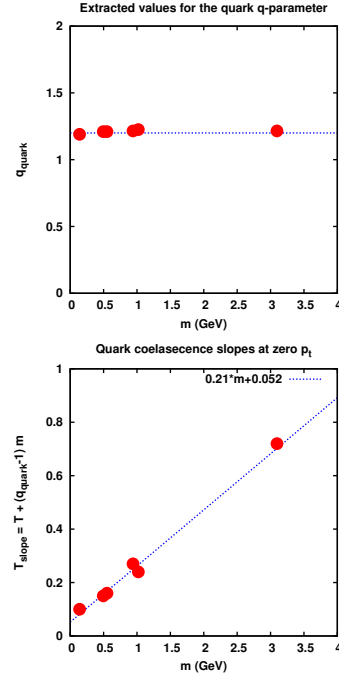


Fig. 10. The q parameter of quark matter extracted from hadronic spectra assuming quark coalescence at a sudden hadron formation (upper picture). The spectral inverse slope as a function of the minimal energy $E_{\text{min}} = m$ agree with the linear prediction from the coalescence scaling.

Furthermore these predictions of the non-extensive phenomenology meet the curves from pQCD calculations, with the following surmised properties of quark matter at RHIC: $T = 140 \dots 180$ MeV, $q = 1.22$, $v_T = 0.6$. [24]. Deviations from the product rule suggested by the simplest quark coalescence idea occur at p_T values lower than 1 GeV. We attribute these to a further constituent in real hadrons, namely non-perturbative gluons simulated by a string energy contribution[26].

We note that a stringy interaction remainder above the color deconfinement temperature $T_c \approx 170$ MeV in

quark gluon plasma also describes the main effects on the quark matter equation of state seen in lattice QCD calculations successfully [27]. Both the presence of a string like pair potential for a however minor percentage of pairs, as a microscopic model, and the assumption of non-extensivity, $q > 1$, as a descriptive phenomenology are able to explain the value of the energy per particle, $E/N = 6T = 1$ GeV, which has been found by fitting statistical hadronic resonance gas models with Boltzmann distributions.

Finally some remarks are in order to the Tsallis-Pareto fits to energy spectra. In several cases naively a fit is done in the original form:

$$f(E) \sim \left(1 + (q-1)\frac{E}{T}\right)^{-\frac{1}{q-1}} \quad (119)$$

to which the following inverse logarithmic slope dependence belongs:

$$T_{\text{slope}} = T + (q-1)E. \quad (120)$$

A more sophisticated approach (as one suggested in [17]), however, uses the original Tsallis-Pareto form for the number density distributions of particles and for the generating thermodynamical potential, for the logarithm of the canonical partition function. This way in this second approach the energy distribution is described by the q -th power of the naive factor:

$$f(E) \sim \left(1 + (\tilde{q}-1)\frac{E}{\tilde{T}}\right)^{-\frac{\tilde{q}}{\tilde{q}-1}} \quad (121)$$

and the corresponding inverse slope

$$T_{\text{slope}} = \frac{1}{\tilde{q}}\tilde{T} + \left(1 - \frac{1}{\tilde{q}}\right)E. \quad (122)$$

We observe that the qualitative behavior is the same, but the interpretation of the fit parameter is different in these different approaches. The correspondence between the energy spectrum fit parameters is given as

$$\begin{aligned} T &= \tilde{T}/\tilde{q}, \\ q &= 2 - 1/\tilde{q}. \end{aligned} \quad (123)$$

In a sense q and \tilde{q} are double-duals of each other, both using the $1/q$ - and the $2-q$ -duality. Also the estimated temperature parameter differ. Typical values from relativistic heavy ion experiments are $q \approx 1.2$ and $\tilde{q} \approx 1.25$, as well as, $\tilde{T} = 1.25T$.

6 Conclusion

In conclusion we reviewed basic concepts of non-extensive thermodynamics which may be relevant in understanding particular features of hadronic spectra stemming from relativistic heavy ion collisions. The overall presence of relativistic speeds of particles in the physical system under investigation on the other hand offers a unique possibility

to study and - whenever necessary - to generalize familiar thermodynamics.

We presented some general arguments for a possible need to face with total energy and entropy not being proportional to the particle number even in the large N limit. These arguments are based on the long range nature of pair interactions. This phenomenon, called non-extensivity, was then related to the generalization of composition rules of the familiar thermodynamical extensives, like energy and entropy. We have mathematically proved that abstract composition rules become symmetric and associative in the large N limit, provided that the composition function, $h(x, y)$, is at least right-sided differentiable at $y = 0^+$. This means that associative composition rules constitute attractors among all rules when approaching the thermodynamical limit. As a consequence the associativity of the composition rule is a thermodynamical requirement.

The key quantity in this proof, the formal logarithm, relates the abstract composition rule to the addition of the system size indicator, to the particle number N . We gave the formula how to construct it. Based on the formal logarithm the widely used *deformed* exponential and logarithm functions can easily be derived. While the former describes the energy distribution in canonical equilibrium, the latter defines a generalized formula for the entropy. An additive entropy can be always gained from this expression by taking its formal logarithm.

We presented some often used composition rules together with the corresponding equilibrium energy distributions and entropy formulas including the traditional Boltzmann-Gibbs formula derived from the simple addition (extensivity), the Tsallis rule, the Kaniadakis rule and - for the sake of demonstration - the Einstein rule for composing relativistic velocities. Our general method in this case leads to the rapidity as the additive formal logarithm. Among non-associative composition rules the class of $h(x, y) = x + y + G(xy)$ is found to be particularly interesting in high energy physics, since it asymptotically approaches the Tsallis rule leading to power-law tailed energy distributions in canonical equilibrium. We have demonstrated that such a composition rule may emerge in the extreme relativistic kinematics limit from an energy correction to a pair of particles in a medium which is a function of the Lorentz invariant relative momentum squared variable Q^2 . In fact this is frequently the case when following several interactions among partons according to the formulas derived from (or at least motivated by) QCD. Finally it is interesting to note that the elementary property, $h(x, 0) = x$ is related to $\sigma(1) = 0$ property of the entropy density function if the composition rule $h(x, y)$ is assumed for composite states with factorizing probabilities. In this case the general result $\sigma(p) = \ln_a(1/p)$ emerges in the thermodynamical limit, with $\ln_a = L^{-1} \circ \ln$ being the corresponding deformed logarithm function.

The - in some sense opposite - requirement, i.e. aiming at an additive entropy formula while the probabilities do not factorize, but their logarithms follow a general composition rule $h(x, y)$ instead of the usual addition, leads

to a more complex relation between the formal logarithm of the rule, L , and the entropy density σ .

In the second part of this review we compiled the most important numerical results on parton cascade simulations of non-extensive systems. Following the presentation of a class of generalized Boltzmann equations, and proving that the second law of thermodynamics can only be fulfilled if the derivative of the entropy density is a linear expression of the deformed logarithm of the one-particle phase space density (cf. eq.(80)), we presented some details of the kinematical description of relativistic particles in such simulations. We paid special attention to the random choice of particle momenta after an, in energy non-additive, pair collision (which can have a physical reason in the influence of third or further particles, or fields in a dense medium).

Results on the phase space evolution under non-extensive energy composition rules were presented demonstrating the ability of such a computer simulation to generate power-law tailed energy spectra in the detailed balance state of the non-extensive Boltzmann equation. The important question of equilibration between two large subsystems, related to the zeroth theorem of thermodynamics, was also investigated by us numerically in this framework. We found that non-extensive systems with the power-law tailed Tsallis-Pareto energy distributions do behave as they should, just the thermodynamic temperature, T is related to the microcanonical equation of state, $S(E)$, by receiving corrections due to the formal logarithms of the entropy and energy composition rules (cf. eq.(109)).

Finally our studies on the hadronization of quark matter in relativistic heavy ion collisions revealed that if the quark coalescence is a dominant mechanism, then the non-extensivity parameter, $q - 1 = aT$ also must show the quark number scaling. This assumption can be and should be tested on experimental data and should be related to other information on quark coalescence, e.g. to those obtained from studies of the elliptic flow.

Certainly there remain open questions for further research. Among them the study of the quark matter equation of state with elementary field theory means, as lattice QCD, in a non-extensive canonical state is still a hard challenge. Also the determination of the pair correlation function, $g(r)$, from first principles in microscopical calculations should help to identify those physical situations where the concepts and formalism of non-extensive thermodynamics have to be used. Meanwhile the physical reason for a non-exponential energy distribution can be numerous. The quark gluon plasma is a wonderful candidate for finding non-extensive behavior, since long range effects are there at any finite temperature.

Acknowledgments

This work has been supported by the Hungarian National Science Fund, OTKA (K49466, K68108). Discussions with C. Tsallis, G. Wilk, T. Kodama, G. Kaniadakis, P. Ván and A. László are gratefully acknowledged.

References

1. Shannon C. E., Bell Syst. Tech. J.**27**(1948)379; *ibid.* **27**(1948)623.
2. Jaynes E. T., Phys. Rev.**160**(1957)171; *ibid.* **168**(1957)620.
3. Rényi A., Acta Math. Acad. Sci. Hung. **10**(1959)193; *Probability Theory*, North Holland, Amsterdam, 1970.
4. Daróczy Z. Aczél J., *On Measures of Information and their Characterization*, Academic Press, New York, 1975.
5. Fisher R. A., *Statistical Methods and Scientific Inference*, Oliver and Boyd, Edinburgh-London, 1959.
6. Ván P., Physica A**365**(2006)28.
7. A. Rényi, *Probability Theory*, North Holland, Amsterdam, 1970; A. Wehrl, Rev.Mod.Phys. **50**, 221, 1978; Z. Daróczy, Inf.Control **16**, 36, 1970; J. Aczél, Z. Daróczy, *On Measures of Information and their Characterization*, Academic Press, New York, 1975.
8. C. Tsallis, J.Stat.Phys. **52**, 50, 1988; Physica A **221**, 277, 1995; Braz.J.Phys. **29**, 1, 1999; P. Prato, C. Tsallis, Phys.Rev.E **60**, 2398, 1999; V. Latora, A. Rapisarda, C. Tsallis, Phys.Rev.E **64**, 056134, 2001; Physica A **305**, 129, 2002.
9. Biro T. S. Purcsel G., Phys. Rev. Lett.**95**(2005)162302.
10. T. S. Biro, Eur. Phys. Lett.**xx**(2008)pp. e-Print: arXiv:0809.4675 [nucl-th]
11. Tsallis C., Gell-Mann M. Sato Y., Proc. Nat. Acad. Sci.**102**(2005)15377.
12. Castillo E., Iglesias A. Ruíz-Cobo R., *Functional Equations in Applied Sciences*, Elsevier, 2005.
13. Tsallis C., J. Stat. Phys.**52**(1988)479.
14. Kaniadakis G., Phys. Rev. E**66**(2002)056125; *ibid.* **72**(2005)036108; Physica A**365**(2006)17.
15. G. Kaniadakis, Physica A **296**, 405, 2001; Phys.Rev.E **66**, 056125, 1, 2002.
16. J. A. S. Lima, R. Silva, A. R. Plastino, Phys.Rev.Lett. **86**, 2938, 2001
17. A. Lavagno, Phys.Lett.A **301**, 13, 2002. Relativistic Nonextensive Thermodynamics,
18. C. Tsallis, E. P. Borges, cond-mat/0301521; C. Tsallis, E. Brigati, cond-mat/0305606; C. Tsallis, Braz.J.Phys. **29**, 1, 1999; A. Plastino, A. R. Plastino, Braz.J.Phys. **29**, 50, 1999.
19. Q. A. Wang, A. Le Méhauté, J.Math.Phys. **43**, 5079, 2002; Q. A. Wang, Chaos, Solitons and Fractals **14**, 765, 2002; Eur.Phys.J.B **26**, 357, 2002.
20. S.Abe, Physica A **300**, 417, 2001; Phys.Rev.E **63**, 061105, 2001.
21. L. Borland, Phys.Rev.E **57**, 6634, 1998; D. H. Zanette, Braz.J.Phys. **29**, 108, 1999; G. Kaniadakis, Phys.Lett.A **283**, 288, 2001.
22. Biro T. S. Purcsel G., Phys. Lett. A**372**(2008)1174.
23. Biro T. S., Levai P. Zimanyi J., J. Phys. G**28**(2002)1561,
24. Biro T. S., Urmossy K. Barnafoldi G. G. , J. Phys. G**35**(2008)044012,
25. Biro T. S. Urmossy K, Eur. Phys. J. ST**155**(2008)1.
26. Biro T. S. Urmossy K, *Contribution to SQM2008, held at Tsinghua University, Beijing, China, Oct. 6-10. 2008*
27. Biro T. S. Cleymans J., Phys. Rev. C**78**(2008)034902.